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A LOCAL EXISTENCE THEOREM FOR A CLASS OF DELAY DIFFERENTIAL EQUATIONS

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Abstract. The goal of this paper is to show that some classes of partial differential functional equations admit a natural formulation as ordinary functional differential equations in infinite dimensional Banach spaces. Moreover, the equations thus obtained are driven by continuous right-hand sides satisfying the compactness assumptions required by the infinite-dimensional version of a Peano-like existence theorem. Two applications, one to a semilinear wave equation with delay and another one to a pseudoparabolic PDE in Mechanics, are included.

1. Introduction

Let X be a Banach space, $\tau \geq 0$, $\mathcal{X} := C([-\tau,0];X)$, $\sigma \in \mathbb{R}$, $f,g\colon [\sigma,+\infty)\times \mathcal{X} \to X$ be two continuous functions, and $\psi \in \mathcal{X}$. If $u \in C([-\tau,+\infty);X)$ and $t \geq 0$, u_t is defined by $u_t(s) = u(t+s)$ for each $s \in [-\tau,0]$.

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We consider the nonlinear delay differential evolution Cauchy problem

$$\begin{cases} u'(t) = f(t, u_t) + g(t, u_t) & \text{if } t \in [\sigma, T], \\ u(t) = \psi(t - \sigma) & \text{if } t \in [\sigma - \tau, \sigma], \end{cases} \tag{1.1}$$

and we prove that, if f is continuous and compact and g is continuous and locally Lipschitz with respect to its second variable, problem (1.1) has at least one local classical solution. It should be noticed that all existence results which will follow can be completed by global existence results under the additional hypothesis that both f and g have sublinear growth. However, we are not going to touch upon here this kind of problems whose proofs, although based on a recent delay-type Gronwall Inequality established by Burlică and Roşu [4], are standard. For the basic results on delay differential equations the interested reader is referred to Driver [6], Halanay [9] and Hale [10].

A non-delayed version (1.1) was considered by Frigon and O'Reagan [8] who obtained a local existence result by using the Krasnosel'skiĭ Fixed Point Theorem [11]. For other existence results concerning delay equations and inclusions see Mitidieri and Vrabie [12] and [13].

We includ two applications of our abstract existence result. First, we show that, surprisingly, a second order semilinear delay wave equation can be rewritten under the form (1.1), where f is continuous and compact and $g \equiv 0$ and so the local existence follows by means of usual finite-dimensional-type ODE's techniques. Here we exploit an idea of the author which has shown to be effective in the non-delayed case. See Vrabie [15, Theorem 10.4.1, p. 243]. Second, we prove that a pseudoparabolic problem arising in Mechanics can be transformed in the form (1.1) with f and g satisfying all the conditions mentioned before. As far as the second application is concerned, it should be noticed that a similar non-delayed problem was considered by Brill [3] who proved how to reduce it to an ordinary differential equation in an infinite-dimensional Banach space.

DEFINITION 1.1. (a) A function $f\colon [\sigma, +\infty) \times \mathcal{X} \to X$ is called *compact* if it maps bounded subsets in $[\sigma, +\infty) \times \mathcal{X}$ into relatively compact subsets in X.

(b) A function $g\colon [\sigma, +\infty) \times \mathcal{X} \to X$ is called *locally Lipschitz with respect to its last argument* if for each $\psi \in \mathcal{X}$ and each $T > \sigma$ there exist $r = r(T, \psi) > 0$ and $L = L_{T,\psi} > 0$ such that

$$\|g(t, u) - g(t, v)\| \le L\|u - v\|, \quad \text{for each } (t, u), (t, v) \in [\sigma, T] \times D_{\mathcal{X}}(\psi, r),$$

where $D_{\mathcal{X}}(\psi, r)$ denotes the closed ball of center ψ and radius r in $\mathcal{X}$.

DEFINITION 1.2. By a *classical solution* of problem (1.1) on $[\sigma - \tau, T]$ we mean a function $u\colon [\sigma - \tau, T] \to X$ which is of class C^1 on $[\sigma, T]$, satisfies $u'(t) = f(t, u_t) + g(t, u_t)$ for each $t \in [\sigma, T]$ and $u(t) = \psi(t - \sigma)$ for each $t \in [\sigma - \tau, \sigma]$. By a *classical solution* of problem (1.1) on $[\sigma, +\infty)$ we mean a function